Pre-stack wavefield approximations

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ABSTRACT

The double-square-root (DSR) relation offers a platform to perform pre-stack imaging using an extended single wavefield that honors the geometrical configuration between sources, receivers, and the image point, or in other words, pre-stack wavefields. Extrapolating such wavefields, nevertheless, suffers from limitations. Chief among them is the singularity associated with horizontally propagating waves. I have devised highly accurate approximations free of such singularities which are highly accurate. Specifically, I use Padé expansions with denominators given by a power series that is an order lower than that of the numerator, and thus, introduce a free variable to balance the series order and normalize the singularity. For the higher-order Padé approximation, the errors are negligible. Additional simplifications, like recasting the DSR formula as a function of scattering angle, allow for a singularity free form that is useful for constant-angle-gather imaging. A dynamic form of this DSR formula can be supported by kinematic evaluations of the scattering angle to provide efficient pre-stack wavefield construction. Applying a similar approximation to the dip angle yields an efficient 1D wave equation with the scattering and dip angles extracted from, for example, DSR ray tracing. Application to the complex Marmousi data set demonstrates that these approximations, although they may provide less than optimal results, allow for efficient and flexible implementations.

INTRODUCTION

The typical layout and design of seismic surveys allow for high redundancy in measuring the earth scattering response (upgoing wavefields), as well as the scattering potential (image), conveniently described by the image point illumination. This redundancy implies data dependency resulting in measurement connection that exists in the data space. Wave extrapolation using the survey-sinking or double-square-root (DSR) formulation of the wave equation (Claerbout, 1985; Popovici, 1996; de Hoop et al., 2003) provides a straightforward connection between the image and the data, free of the crosscorrelation imaging condition step. A limitation of the DSR conventional implementation is the one-way nature (in depth) of the wave-extrapolation process, which limits the imaging accuracy at large structural dips. An alternative is to extend the survey-sinking approach to extrapolation in time rather than depth in the full source and receiver wavefield (Alkhalifah and Fomel, 2010). Application of two-way extrapolators to modeling and migration follows the exploding reflector concept (Loewenthal et al., 1976; Claerbout, 1985) and allows for upgoing as well as downgoing wavefields. It also provides us with subsurface offsets (or angle gathers) naturally (Fomel, 2011). However, a major hindrance in time extrapolating the DSR equation is an inherent singularity for horizontally traveling waves (Biondi, 2002; Duchkov and de Hoop, 2009; Alkhalifah and Fomel, 2010). Specifically, the singularities in this extended domain wave formulation prevent us from conventionally solving the dynamic form of the DSR equation in the space domain using the (usually more efficient for inhomogeneous media) finite-difference approach.

In this paper, I investigate this singularity and explore ways to circumvent it. I develop approximations to the DSR formula that are free of such singularities, and explore the prospect of solving these approximations in this extended domain by relaxing the accuracy requirements in the offset direction. This includes utilizing the geometrical description of waves (traveltimes and rays) to constrain the offset part of the solution. Additional constraints on dip angle yield simple and even more efficient approximate formulations. I apply the new formulations on the Marmousi data set to expose their strength and weaknesses.

THEORY

Consider a seismic survey \( P(t, s, r, z) \) as a function of time \( t \) and source and receiver locations \( s \) and \( r \) at the surface at depth \( z \). Our goal is to extrapolate the four-dimensional (six-dimensional in 3D) wavefield \( P(t, s, r, z) \) in time. Let \( x \) represent the space
coordinates $x = \{s, r, z\}$, for the 2D case. The wave-extrapolation operator, valid for small $\Delta t$, is (Wards et al., 2008)

$$P(t + \Delta t, x) + P(t - \Delta t, x)$$

$$\approx 2 \int \hat{P}(t, \mathbf{k}) \cos [\phi_0(x, \mathbf{k}) \Delta t] e^{i \mathbf{k} \cdot \mathbf{x}} d\mathbf{k}, \quad (1)$$

where $\hat{P}(t, \mathbf{k})$ represents the prestack wavefield in the wavenumber domain given by $k = \{k_s, k_r, k_h\}$ for 2D media. In the geometrical (high-frequency) approximation, the function $\phi_0(x, \mathbf{k})$ appearing in equation 1 should satisfy the appropriate eikonal equation, which is, in the case of prestack data, the DSR equation. For isotropic media, it has the following form (Alkhalifah and Fomel, 2010):

$$\phi_0 = \frac{v_s^2 v_r^2 [(k_s + k_r)^2 + k_h^2][(k_s - k_r)^2 + k_h^2]}{v_s^2 k_r^2 - v_r^2 (k_r^2 - k_h^2) + 2D}, \quad (2)$$

where

$$D = \sqrt{k_r^2 [v_s^2 v_r^2 (k_s^2 + k_r^2) - v_s^4 k_r^2 - v_r^4 k_h^2]}, \quad (3)$$

and we use the shortened notation $v_2 = v(\xi, \zeta)$, for $\xi = \{s, r\}$, $v_s^2 = v_s^2 + v_r^2$, $v_r^2 = v_s^2 - v_r^2$.

In the $v(z)$ case, $v_r = 2v_s$, and equation 2 reduces to

$$\phi_0 = \frac{v_s^2 [(k_s + k_r)^2 + k_h^2][(k_s - k_r)^2 + k_h^2]}{k_h^2}, \quad (4)$$

which clearly includes the $k_r = 0$ singularity.

The phase operator in equation 2 reduces to our familiar eikonal for waves emanating from a source when we set $k_s = k_r$ and $v = v_i = v_r$. Specifically, defining the half-offset slowness, $k_h = k_r - |k_s|$, and the midpoint slowness, $k_m = k_r + |k_s|$, equation 2 reduces to $k_h^2 + k_m^2 = \phi_0/v_s^2$ when $k_h = 0$ and $v_r = 0$, which is free of singularities.

**THE $v(z)$ FORMULA**

The laterally homogeneous-based phase operator, equation 4, has only one singularity given by $k_r = 0$, which corresponds to horizontally traveling waves. Despite the fact that it is based on considering $v_r = v_s$, we can easily allow the velocity in equation 4 to vary laterally. This corresponds to an approximation in the velocity treatment mainly to the offset component. For many situations, this approximation is reasonable, especially for short offsets. Popovici (1996) demonstrates that such an approximation using a split-step implementation is good enough for the Marmousi data set. Thus, the $v(z)$ assumption considered here is only applied to the offset axis, as we consider the velocity for a particular offset given by the midpoint velocity.

The $k_r = 0$ singularity is an artificial one caused by constraining the source and receiver wavefields to the same vertical wavenumber. The true solution for the phase operator for $k_r = 0$ is given by $v|k_h|$ for $|k_h| > |k_s|$ and by $v|k_s|$ for $|k_h| < |k_s|$. These represent the flanks of the pyramid in our offset-midpoint traveltime formulation, especially as the scatterer is near the surface ($k_r \approx 0$). To enforce this solution, I first expand equation 4 as a function of $k_h$ around its zero value using Padé approximation with a fourth-order numerator and second-order denominator, and obtain

$$\phi_0 \approx \frac{v(8k_s^2 k_r^2 + k_r^4 + 8k_s^4)\sqrt{k_s^2 + k_h^2}}{4k_h^2 (k_h^2 + 2k_r^2)}. \quad (5)$$

The uneven order of the Padé expansion allows us to add a $k_h^2$ term in the denominator with a coefficient that can be used to force the solution to approach the exact one for $k_h = 0$, or asymptotically with respect to the horizontal wavenumbers. Thus,

$$\phi_0 = \frac{v(8k_s^2 k_r^2 + k_r^4 + 8k_s^4)\sqrt{k_s^2 + k_h^2}}{4k_h^2 (k_h^2 + 2k_r^2) + Ak_h^4}. \quad (6)$$

If I set $k_r = 0$, then $A = 1$ yields the proper solution for horizontal reflectors for $k_h < k_r$ and $A = |k_s/k_h|$ for $k_h > k_r$.

A similar approximation can be obtained for a higher-order Padé expansion given by an eighth-order numerator and sixth-order denominator with adding the $Ak_h^4$ term to the denominator to obtain

$$\phi_0 \approx \frac{v(32k_r^6 k_h^2 + 160k_r^4 k_h^4 + 256k_r^2 k_h^6 + k_h^8 + 128k_h^8)\sqrt{k_s^2 + k_h^2}}{8k_h^2 (10k_s^4 k_h^2 + 24k_r^4 k_h^4 + k_h^8 + 16k_h^8) + Ak_h^4}. \quad (7)$$

for $k_h < k_r$, and

$$\phi_0 \approx \frac{v(32k_r^6 k_h^2 + 160k_r^4 k_h^4 + 256k_r^2 k_h^6 + k_h^8 + 128k_h^8)\sqrt{k_s^2 + k_h^2}}{8k_h^2 (10k_s^4 k_h^2 + 24k_r^4 k_h^4 + k_h^8 + 16k_h^8) + Ak_h^4}. \quad (8)$$

for $k_h > k_r$.

The difference between the two new phase extrapolation operators and the original $v(z)$ equation 4 are displayed in Figure 1. For a reasonable range of wavenumbers, the difference is less than 1% for equation 6 and less than 0.01% for equations 7 and 8. Of course, for the large wavenumbers, we are effectively approaching the singularity and the difference between the equations are due to the adjustment we added to normalize that artificial singularity.

**THE DSR-BASED PARTIAL DIFFERENTIAL EQUATION**

To derive a partial differential equation capable of representing the behavior of waves in the extended source-receiver domain with kinematics described by the DSR equation, we cast the DSR equation (equation 2) in a polynomial form, set $\omega = \phi_0$ as follows:

$$2\omega^2 v_r^2 v_s^2 (v_r^2 - k_s^2 - k_h^2) + k_h^2 (v_r - v_s)(v_r + v_s)$$

$$+ v_s^2 (k_s - k_h)(k_s + k_h) + v_s^4 v_r^4 (2k_s^2 - k_h^2)$$

$$+ k_s^4 (k_s^2 + k_h^2)^2 + \omega^4 (v_r^2 - v_s^2)^2 = 0. \quad (9)$$

As a reminder, $v_s$ and $v_r$ are the source and receiver velocities, respectively, and can be represented as $v(s, z)$ and $v(r, z)$, because the velocity for the medium is a single function. Multiplying both sides of equation 9 by the wavefield in the Fourier domain, $F(k_s, k_r, k_h, \omega)$, as well as using inverse Fourier transform on $k_s$, $k_r$, and $k_h (k_h \rightarrow -i(d/ds), k_r \rightarrow -i(d/dr)$, and $k_h \rightarrow -i(d/dz)$).
as well as inverse Fourier transform on \( \omega \) \((\omega \rightarrow i(\partial/\partial t))\), yields a wave equation in the space-time domain, given by

\[
\left( v_0^2 - v_2^2 \right) \frac{\partial^4 F}{\partial t^4} = 2(v_0^2 v_2^2 - v_4^2 v_2^2) \left( \frac{\partial^4 F}{\partial t^2 \partial x^2} - \frac{\partial^4 F}{\partial t^2 \partial z^2} \right) + 2(v_0^4 v_2^4 + v_4^4 v_2^4) \frac{\partial^4 F}{\partial t^2 \partial x^2} - v_0^4 v_2^4 \times \left( \frac{\partial^4 F}{\partial x^4} - 2 \frac{\partial^4 F}{\partial x^2 \partial z^2} + \frac{\partial^4 F}{\partial x^2 \partial z^2} - 2 \frac{\partial^4 F}{\partial t^2 \partial z^2} + 2 \frac{\partial^4 F}{\partial z^4} + \frac{\partial^4 F}{\partial z^4} \right).
\]

(10)

Unlike in the conventional wave equation for 2D media which is second-order in time, equation 10 is fourth-order in time, and thus can provide us up to four independent solutions. Two of these solutions correspond to the actual incoming and outgoing P-wave prestack wavefields. The other two solutions are artifacts of the development and specifically of squaring the DSR square roots. In addition, as we extrapolate the solution in time, the equation has a regular singularity when the velocity for the source and receiver are equal as the equation reduces to second-order in time. As a result, solving this partial differential equation using finite-difference approximations yields unstable results.

**SCATTERING ANGLE-BASED FORMULATION**

Considering that a convenient and useful representation of the offset axis is given by the scattering (reflection) angle, I transform the half-offset wavenumber in equation 9 to the reflection angle using \( \psi_0 = k_2 \tan \theta \). In addition, by substituting \( v_0^2 = v_2^2 - v_2^2 \) and \( v_2^2 = v_2^2 + v_2^2 \), we end up with the following:

\[
\begin{align*}
&\frac{\partial}{\partial x^4} - 4(v_0^4 - v_2^4) \frac{\partial^3 F}{\partial x^2 \partial z^2} + (v_0^4 - v_2^4)^2 \left[ (k_2^4 + k_2^2 \tan^2(\theta) + 2k_2^2)^2 - (k_2^4 + k_2^2 \tan^2(\theta))^2 \right] = 0. \\
\end{align*}
\]

(11)

In this form, we are either considering \( \theta \) to be constant, or we need to evaluate \( \theta \) using other means. A constant \( \theta \) along the wavefield might be valid for small offsets. Considering the relatively smaller scattering angles (limited by offset) that we deal with, compared with dips, this approximation is expected to work in many places. A more appropriate representation is given by a constant \( k_2 \), which is valid for a vertically inhomogeneous medium approximation in the offset axis. However, a representation in \( k_2 \), unlike that in \( \theta \), does not remove the \( k_2 = 0 \) singularity in equation 9. Setting \( v_2 = 0 \) in equation 11 yields the following easy form (Alkhalifah, 2012):

\[
v_0^2 k_2^4 + v_2^2 k_2^4 = \cos(\theta)^2 \omega^2.
\]

(12)

which is free of singularities. It introduces a \( \cos \theta \) weighting function to the velocity. It also provides a plane-wave representation to the offset axis, which may be convenient. The dynamic form of this equation, obtained by introducing a wavefield in the Fourier domain followed by inverse Fourier transforms, is given by

\[
\cos(\theta)^2 \frac{\partial^2 F}{\partial t^2} = v_2^4 \left( \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial z^2} \right).
\]

(13)

Of course, this equation is a DSR formulation simplified by considering a \( v(\theta) \) medium along the offset axis. Moreover, it includes a plane-wave description of the offset axis given by the scattering angle, \( \theta \). This equation can be used for prestack exploding reflector modeling (or imaging) applied to constant scattering angles. However, even in \( v(\theta) \) media, the scattering angle changes with depth \( (p_0 = \text{constant}) \). Thus, we can use \( v(\theta) \) plane-wave ray tracing along the offset axis to obtain an approximate \( \theta(z, \theta) \) field per mid-point, however, for a fixed reflector dip. Utilizing high-frequency asymptotics to constrain the wavefields along the offset axis should not hamper its applicability to complex media.

**DIP AND SCATTERING ANGLE**

Likewise, the dip angle \( \psi \) satisfies \( \tan \psi = k_3/k_2 \) in the DSR formulation, and thus, equation 12 yields an even easier form (Alkhalifah, 2012),

\[
v_0^2 k_2^4 = \cos(\theta)^2 \cos(\psi)^2 \omega^2.
\]

(14)

Now, this additional \( \cos \psi \) factor corresponds actually to dip. In homogeneous media, the normal moveout velocity for a dipping reflector is given by \( v_0/\cos(\psi) \) (Levin, 1971). A dynamic form of equation 14 is given by

\[
\cos(\theta)^2 \cos(\psi)^2 \frac{\partial^2 P}{\partial t^2} = v_2^2 \frac{\partial^2 P}{\partial x^2}.
\]

(15)

This 1D wave equation is valid for plane-wave propagation in the dip and scattering angle. Layered media plane-wave ray tracing can be used to obtain maps of \( \theta(z, \psi) \) and \( \psi(z, \theta) \) per location (image point). More exotic ray tracing can be employed to handle more complex media. The offset and midpoint information are embedded in the scattering and dip angles, respectively. Despite the ray tracing, we are still extrapolating prestack wavefields using equation 15, however, approximately.

**THE IMPLEMENTATION**

Marrying high-frequency asymptotic-based ray tracing and wavefield extrapolation is always a challenge; however, the DSR formulation provides a suitable platform to do that as it explicitly
provides the required geometrical information concerning the source-receiver-image point configuration. At first glance, the required ray tracing, per image point, seems exhaustive. However, phase space escape traveltime methods (Fomel and Sethian, 2002) are appropriate for the implementation of this approach. They provide traveltime and angle information for all possible source locations from every image point in one sweep so angle maps can be readily extracted from the solution.

To use equation 13 for imaging purposes, we first need to slant stack the input prestack data along the offset axis. This will allow us to extrapolate separate constant angle (or offset-ray parameter) gathers backward in time using reflection angle tracing (see Appendix A) to evaluate $\theta$ as a function of $x$ and $z$. However, we need also to decompose the data to localized plane-wave panels in midpoint as dip information is also required to trace the reflection angle gather.

I extrapolate prestack wavefields using equation 13 in a homogeneous medium to observe some of the prestack wavefield features. Starting with an initial wavefield given by the image of the French model (French, 1974) in the angle domain (Figure 2a), I show snapshots of the wavefield at different times in Figures 2b, and 2d. Clearly, the prestack wave is propagating upward and downward courtesy of the time extrapolation implementation. For larger angle gathers, the waves move faster as they experience higher effective velocity in the plane-wave domain. The 2D nature of our extrapolation operator given by equation 13 allows for separate angle section extrapolations, and significant reduction in cost.

The data recorded on the surface are given in Figure 3. Instead of the offset axis, the data are evaluated as a function of reflection angle (in this case offset-ray parameter). A simple inverse slant-stack transform can take us to offsets (Figure 4), which is the default axis provided by the DSR formulation. The data include all the reflection and diffraction events expected for such a model. Thus, despite the flexibility, which now allows us to extrapolate constant scattering angles separately for this homogeneous model case, the data are apparently complete.

The Marmousi model

The Marmousi model and data set (Versteeg, 1993) have long served the imaging community as the premier test example for newly developed 2D imaging algorithms. The target zone is a reservoir located at a depth of about 2500 m under a complex overburden. The model (Figure 5) includes strong velocity variations in the lateral and the vertical directions (with a minimum velocity of 1500 m/s and a maximum of 5500 m/s). I use the model to generate the synthetic data set using a conventional fourth-order in space and second order in time finite-difference approximation to the wave equation. The modeling parameters are $\Delta t = 0.4$ ms, $\Delta x = \Delta z = 4$ m and a Ricker wavelet source with peak frequency of 30 Hz. The shot interval is 8 m, and the receiver range satisfies

$$-2512 \text{ m} \leq r - s \leq 2512 \text{ m}$$

for each shot. The receiver spacing is 16 m.

I first use the Marmousi model to test approximations 5 and 6, which are free of the $k_z$ singularity, and compare them with previous results (Alkhalifah and Fomel, 2010) based on the exact DSR
formulation. I inject in reverse (time) the Marmousi prestack data set at the surface forming the zero depth boundary of the prestack domain and extrapolate the 3D wavefield, that includes subsurface offset, in time to zero (the imaging condition) using the new approximations. Snapshots of the wavefield are shown in Figure 6. The snapshots display the evolution of the prestack wavefield as it starts to propagate from the surface to the zero offset plane (the image plane). Despite the approximations used in deriving the new free-of-singularity extrapolator operator, the resultant image given by the zero-offset plane is fairly good (Figure 7a). It clearly imaged the assumed reservoir structure and it is generally cleaner than the one obtained using the original formulation given by equation 2 (Figure 7b). The difference is especially obvious along the steeply dipping fault under lateral location 6200 m.

We can also image the Marmousi using the much faster 2D formula, equation 13, which allows for separate constant reflection angle gather treatment. However, to compute the reflection angle needed by the formula, for simplicity, I assume that the reflectors are horizontal and velocity does not vary laterally, rendering simple ray-tracing equations to compute the reflection angle as a function of depth and lateral position. Unlike the original equations given as a function of subsurface offset, the formula here does not allow energy to propagate from one reflection angle to another, and thus the reflection angle axis corresponds to the surface one. This is synonymous to constant-offset Kirchhoff migration. At zero time, an accurate imaging process and velocity model should render flat moveouts and based on the plane-wave nature of the offset axis, we only need to sum the flat angle component to obtain the image (Figure 8). That image is displayed in Figure 9, and despite the $r(z)$ and zero-dip approximations employed in estimating the reflection angle, the image looks reasonable and more importantly, very cheap to obtain. Specifically, this implementation allows for a zero-offset exploding reflector migration cost and simplicity for each common scattering angle gather. Compared to RTM requiring two extrapolations for each shot gather, the cost savings with this new approximation is high. The accuracy can be improved with more accurate ray tracing corresponding to the actual apparent dip of reflections.

DISCUSSION

The prestack wavefield extrapolation suggested here and based on equations 13 and 15 requires knowledge of the scattering and dip angles, at least approximately, and possibly by tracing these angles (see Appendix A). Specifically, the wavefield is represented by the plane-wave components along the offset axis for equation 13 and along the midpoint axis as well, for equation 15. In the case of equation 13, despite the fact that only the reflection angle information is required, such reflection angles depend on dip, and thus, reflector dip information or at least the reflection slope is also needed. This suggests implementations synonymous to fast-beam methods (Gao et al., 2006), in which the wavefield is locally slant stacked along the midpoint axis and only the most energetic of plane waves are extrapolated. Thus, like fast-beams methods, these equations could allow for crude and fast extrapolation of wavefields by using data slope information.

Another way to look at this new representation, is to consider the scattering (reflection) angle and possibly the dip angle as
degrees of freedom to help improve the imaging process. Specifically, the reflection angle (as a replacement to offset) is the key velocity discrimination variable. This idea has routes in the common-reflection-stack (CRS) implementation (Jager et al., 2001); however, with a dynamic flavor and with respect to the image domain, instead of the data domain. For an accurate velocity model, the image should be consistent over reflection angles displaying a flat image signature over scattering angle. In equation 13, the velocity and reflection angle can be combined in a single (velocity-like) variable as a function of reflection angle. Symes (2008) suggests adding a dimension to the velocity model that will allow us to integrate full-waveform inversion with differential semblance analysis migration velocity analysis. The formulation in this paper allows for such a treatment naturally, but approximately. For example, we can define a variable that combines the velocity and reflection angle, \( \tilde{v}(x, \theta) = v(x)/\cos \theta \), and invert for this variable directly. An accurate image is defined by a differential semblance condition in which \( v(x) = \tilde{v}(x, \theta) \cos \theta \) does not vary with reflection angle. Though \( \theta \) may vary with depth for an image point, we define constant-\( \theta \) gathers by their actual value at the image point. This is similar to constant-offset Kirchhoff migration in which the offset corresponds to that on the surface. With constant angle gathers, it makes more sense to model from image to data, which provides opportunities in least-squares migration and inversion.

Nevertheless, all the above potential uses are introduced within the initial intention of the paper of providing approximations. In many applications, approximations provide insights into the more accurate and general formulation, but it also serves to make methods more practical to implement, granted we recognize the nature of the approximation. Thus, the prestack wavefield approximations introduced here are meant to alleviate the singularity associated with the DSR formulation. In equations 13 and 15, the singularities are alleviated by introducing additional assumptions that renders the prestack wavefield accurate for homogeneous, and \( v(z) \) media, but approximate for generally inhomogeneous media. The level of approximation depends on how the reflection angle \( \theta \) is evaluated.

Figure 6. Snapshots of the imaging process of the wavefield in the prestack domain starting with time equal 2.4 s (a) to time equal 0 s (f) in an exploding reflector modeling application. The extrapolation operator is based on the \( v(z) \) approximation 6. The solid lines cutting through each of the sections of the three dimensions represent the locations of the slices of the 3D volume.
CONCLUSIONS

Combining ray tracing and wavefield extrapolation to speed up wavefield construction and seismic imaging is always a challenge. However, the DSR formulation provides a natural platform to do that as it explicitly provides geometrical information on the source-receiver-image point configuration. Thus, I develop DSR formulations that are capable of extrapolating wavefields in the pre-stack domain and are free of singularities. They are based on Padé approximations that are highly accurate. I also recast the DSR formulation in terms of reflection and dip angles allowing for simplified approximate prestack wavefield extrapolation operators free of singularities. The reflection and dip angles can be computed using DSR ray tracing. For even a horizontal reflector simplification in computing the reflection angles in a $v(z)$ medium, these approximations provide reasonable results for the Marmousi data set. They specifically provide large reduction in imaging cost with a flexible separate scattering-angle implementation suitable for velocity analysis applications.

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APPENDIX A

ANGLE TRACING

The scattering and the dip angles can be traced along the prestack isochron rays. I first define the ray parameters (slowness vector) in the prestack domain given by $p_s = k_s/\omega$ for the midpoint coordinate, $p_h = k_h/\omega$ for the offset coordinate, and $p_z = k_z/\omega$ for the depth coordinate in 2D media. Based on the laterally homogenous version of the DSR Hamilton-Jacobi equation...
the isochron rays (Duchkov and de Hoop, 2009), as a function of a monotonically increasing variable \( s \), satisfy the following first-order ordinary differential equations:

\[
\frac{dp_x}{ds} = (p_x^2 + p_z^2)(p_h^2 + p_z^2) \frac{dv^2}{dx}, \quad \text{(A-2)}
\]

\[
\frac{dp_z}{ds} = (p_x^2 + p_z^2)(p_h^2 + p_z^2) \frac{dv^2}{dz}, \quad \text{(A-3)}
\]

\[
\frac{dp_h}{ds} = 0, \quad \text{(A-4)}
\]

\[
\frac{dx}{ds} = 2p_x(p_h^2 + p_z^2)v^2(x, z), \quad \text{(A-5)}
\]

\[
\frac{dz}{ds} = 2p_z((p_x^2 + p_h^2 + 2p_z^2)v^2(x, z) - 1), \quad \text{(A-6)}
\]

\[
\frac{dh}{ds} = 2p_h(p_x^2 + p_z^2)v^2(x, z), \quad \text{(A-7)}
\]

\[
\frac{dt}{ds} = p_x \frac{dx}{ds} + p_h \frac{dh}{ds} + p_z \frac{dz}{ds} = 2(p_x p_h + p_z^2)(p_h^2 + p_z^2)v^2(x, z) - 2p_z^2. \quad \text{(A-8)}
\]

Because we are using the \( v(z) \) assumption for the offset coordinate, \( dp_h/ds = 0 \). Using equation A-8, we can formulate the ray-tracing equations as a function of traveltime resulting in the following set of ordinary differential equations:

\[
\frac{dp_x}{dt} = \frac{(p_x^2 + p_z^2)(p_h^2 + p_z^2) \frac{dv^2}{dx}}{2(p_x p_h + p_z^2)(p_x^2 + p_h^2 + 2p_z^2)v^2(x, z) - 2p_z^2}, \quad \text{(A-9)}
\]

\[
\frac{dp_z}{dt} = \frac{(p_x^2 + p_z^2)(p_h^2 + p_z^2) \frac{dv^2}{dz}}{2(p_x p_h + p_z^2)(p_x^2 + p_h^2 + 2p_z^2)v^2(x, z) - 2p_z^2}, \quad \text{(A-10)}
\]

\[
\frac{dp_h}{dt} = 0, \quad \text{(A-11)}
\]

\[
\frac{dx}{dt} = \frac{2p_x(p_h^2 + p_z^2)v^2(x, z)}{2(p_x p_h + p_z^2)(p_x^2 + p_h^2 + 2p_z^2)v^2(x, z) - 2p_z^2}, \quad \text{(A-12)}
\]

\[
\frac{dz}{dt} = \frac{2p_z((p_x^2 + p_h^2 + 2p_z^2)v^2(x, z) - 1)}{2(p_x p_h + p_z^2)(p_x^2 + p_h^2 + 2p_z^2)v^2(x, z) - 2p_z^2}, \quad \text{(A-13)}
\]

\[
\frac{dh}{dt} = \frac{2p_h(p_x^2 + p_z^2)v^2(x, z)}{2(p_x p_h + p_z^2)(p_x^2 + p_h^2 + 2p_z^2)v^2(x, z) - 2p_z^2}, \quad \text{(A-14)}
\]

The scattering angle, \( \tan \theta = p_h/p_z \), along the ray satisfies

\[
\frac{dt}{d\theta} = \frac{\tan \theta dp_h}{dp_h} + \frac{\tan \theta dp_z}{dp_z}. \quad \text{(A-15)}
\]

Given that \( \tan \theta/\partial p_h = 1/p_z \) and \( \tan \theta/\partial p_z = -p_h/p_z^2 \) then,

\[
\frac{dt}{d\theta} = \frac{p_h(p_x^2 + p_z^2)(p_x^2 + p_h^2 + 2p_z^2) \frac{dv^2}{dz}}{2(p_x p_h + p_z^2)(p_x^2 + p_h^2 + 2p_z^2)v^2(x, z) - 2p_z^2}. \quad \text{(A-16)}
\]

Similarly, the dip angle, \( \phi = p_x/p_z \), along the ray satisfies

\[
\frac{dt}{d\phi} = \frac{\tan \phi dp_x}{dp_x} + \frac{\tan \phi dp_z}{dp_z}. \quad \text{(A-17)}
\]

Given that \( \tan \phi/\partial p_x = 1/p_z \) and \( \tan \phi/\partial p_z = -p_x/p_z^2 \) then,

\[
\frac{dt}{d\phi} = \frac{(p_x^2 + p_z^2)(p_x^2 + p_z^2) \frac{dv^2}{dx}}{2(p_x p_h + p_z^2)(p_x^2 + p_h^2 + 2p_z^2)v^2(x, z) - 2p_z^2}. \quad \text{(A-18)}
\]

For a laterally homogenous medium, the scattering angle remains stationary along the ray. On the other hand, the dip angle remains stationary only for horizontal reflectors.

**REFERENCES**


