Acoustic anisotropic wavefields through perturbation theory

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ABSTRACT

Solving the anisotropic acoustic wave equation numerically using finite-difference methods introduces many problems and media restriction requirements, and it rarely contributes to the ability to resolve the anisotropy parameters. Among these restrictions are the inability to handle media with $\eta < 0$ and the presence of shear-wave artifacts in the solution. Both limitations do not exist in the solution of the elliptical anisotropic acoustic wave equation. Using perturbation theory in developing the solution of the anisotropic acoustic wave equation allows direct access to the desired limitation-free solutions, that is, solutions perturbed from the elliptical anisotropic background medium. It also provides a platform for parameter estimation because of the ability to isolate the wavefield dependency on the perturbed anisotropy parameters. As a result, I derive partial differential equations that relate changes in the wavefield to perturbations in the anisotropy parameters. The solutions of the perturbation equations represent the coefficients of a Taylor-series-type expansion of the wavefield as a function of the perturbed parameter, which is in this case $\eta$ or the tilt of the symmetry axis. The expansion with respect to the symmetry axis allows use of an acoustic transversely isotropic media with a vertical symmetry axis (VTI) kernel to estimate the background wavefield and the corresponding perturbation coefficients. The VTI extrapolation kernel is about one-fourth the cost of the transversely isotropic model with a tilt in the symmetry axis kernel. Thus, for a small symmetry axis tilt, the cost of migration using a first-order expansion can be reduced. The effectiveness of the approach was demonstrated on the Marmousi model.

INTRODUCTION

The acoustic anisotropic wave equation is now heavily used in industry as anisotropy inclusion in imaging the subsurface is becoming a necessity. The necessity stems from the fact that anisotropic migrations have recently yielded superior images to those attained using the conventional isotropic assumption of the earth subsurface, even in complex media (Zhou et al., 2004; Huang et al., 2008). This preference for anisotropic imaging is practiced in spite of the many limitations associated with solving the anisotropic acoustic wave equation (Alkhalifah, 2000). These limitations include constraints on the anisotropic models, specifically that $\eta \leq 0$. Also, in its conventional implementation, wavefields extracted from the fourth-order anisotropic acoustic wave equation include shear-wave artifacts. It is also a fact that solving the anisotropic acoustic wave equation costs more than solving its isotropic counterpart, and the cost for the tilted anisotropic case could be four times higher.

These limitations pale in comparison with the challenges involved in estimating the anisotropy parameters. It is hard enough to resolve a velocity model that may change vertically and laterally on a fine scale — imagine doing so for more than one velocity (or parameter) and specifically resolving up to five parameters in the widely used transversely isotropic (TI) model with a tilt in the symmetry axis (TTI) in 3D. Despite the many approximations used to simplify the anisotropic description, including weak anisotropy approximations of the polarization and phase angles (Pšenčík and Gajewski, 1998), parameter estimation remains difficult in complex media. A dip-oriented TI assumption reduces the required parameters to three (Alkhalifah and Sava, 2010). Nevertheless, resolving even three parameters is at least three times more difficult.

The first-order dependence of wavefields, and specifically the acoustic wave equation, on media parameters is described by the Born approximation. It is a single scattering approximation, used in seismic applications to approximate the perturbed wavefield...
due to small perturbations in the reference medium. It is also, in its inverse form, used to help infer medium parameters from observed wavefields (Cohen and Bleistein, 1977; Panning et al., 2009). In the spirit of the Born approximation, the partial differential equations used here describe the wavefield first- and second-order dependence on perturbations in anisotropy parameters from a background isotropic or elliptically anisotropic medium. This type of formulation allows isolation of the anisotropy influence even in complex media, and thus, it provides the opportunity to measure this influence.

In acoustic TI media with a vertical symmetry axis (VTI), Alkhalifah (1998) demonstrated that a new representation in terms of just three parameters is sufficient to solve for the wavefield. These three parameters are the P-wave vertical velocity (or symmetry-axis direction) \( v_p \), the normal-moveout velocity for a horizontal reflector

\[
v = v_p \sqrt{1 + 2\delta},
\]

and the anisotropy coefficient

\[
\eta \equiv 0.5 \left( \frac{v_h^2}{v_p^2} - 1 \right) = \frac{\epsilon - \delta}{1 + 2\delta},
\]

where \( v_p \) is the horizontal velocity and \( \epsilon \) and \( \delta \) are Thomsen (1986) parameters.

Alkhalifah (2000) introduces the acoustic wave equation as an alternative to the elastic wave equation. It produces a very similar description of the P-wave kinematics including the geometrical amplitude behavior to that of the elastic solution, but at a reduced cost and effort. It is especially useful for imaging applications because we tend to condition the data to comply with the acoustic assumption. Following in the footsteps of Alkhalifah (2000), I apply perturbation theory to the acoustic anisotropic wave equation in hope of alleviating some of the aforementioned weaknesses of the original equation. I focus on perturbing from a background elliptical anisotropic model as solving the wave equation for such a model is free of all the limitations associated with the VTI equation. This implies that I develop solutions as a function of a polynomial series in terms of powers in \( \eta \), and also the tilt. Despite the constant \( \eta \) and tilt assumption, the solution should be valid for expansion parameter changes causing wavelength perturbations not exceeding the seismic wavelength. I demonstrate the usefulness of the approach on synthetic data, including the Marmousi model.

**PERTURBATION THEORY**

The general procedure of the perturbation method is to identify a small parameter (i.e., tilt angle or \( \eta \)), such that when this parameter is set to zero the problem (for example, the differential equation) becomes easily soluble. Thematically, the approach decomposes a tough problem into an infinite number of relatively easy ones. Hence, the perturbation method is most useful when the first few steps reveal the important features of the solution and the remaining ones give small corrections. Another feature of the perturbation method is that it does not add additional solutions to the solutions already obtained from the unperturbed medium (i.e., background medium). The method perturbs those solutions obtained for the background medium (in this case, elliptical anisotropic or VTI) to accommodate the perturbation in the medium.

To approach the TTI problem, two routes can be taken. Compute a VTI background wavefield solution and perturb from that solution as a function of tilt angle to approximate the TTI solution. One advantage to this approach is the relatively low cost of the VTI solution, and it allows me to readily invert for a smooth tilt angle field. However, accuracy will depend directly on the size of the tilt angle. Alternatively, I compute a tilted elliptical anisotropic solution and perturb that solution in terms of \( \eta \) to approximate the TTI behavior. A feature of this approach is the simplicity of the elliptical wave equation (second order) with solutions free of shear-wave artifacts. In the following, I develop both approaches.

**Perturbation in tilt angle**

For a TTI medium, the derivatives in the acoustic VTI wave equation (Alkhalifah, 2000) are taken with respect to the symmetry axis direction, and thus, I have to rotate the derivatives using the following Jacobian in 3D:

\[
\begin{pmatrix}
\cos \phi & \sin \phi & \sin \theta \\
-\sin \phi & \cos \phi & 0 \\
-\cos \phi & \sin \theta & -\sin \phi \cos \theta & \cos \phi \\
\end{pmatrix},
\]

to obtain a wave equation corresponding to the conventional computational coordinates governed by the acquisition surface. In equation 3, \( \theta \) is the angle of the symmetry axis measured from the vertical and \( \phi \) corresponds to the azimuth of the vertical plane that contains the symmetry axis measured from the x-axis (typically the inline direction).

The TTI (or VTI) acoustic wave equation, despite its fourth-order nature, can be described by a pair of second-order linear partial differential equations, and specifically I use the form of Zhang et al. (2011) given by

\[
\frac{\partial^2}{\partial z^2} p + v_p^2 H_2[p] + v_f^2 H_1[q] = 0,
\]

Here, \( p \) and \( q \) are two wavefields that depend on space \( x \) and time \( t \) and \( H_1 \) and \( H_2 \) are differential operators applied to the wavefields in square brackets. They include rotation to honor the tilt in the axis of symmetry. In VTI media, \( H_1 \) and \( H_2 \) reduce to \( \partial^2/\partial x^2 \) and \( \partial^2/\partial y^2 \) respectively. To honor tilt, I rotate these derivatives using the Jacobian (equation 3) (Zhang et al., 2011). The velocities \( v_p \), \( v_f \), and \( v_r \) are those along the symmetry axis, orthogonal to the symmetry axis and “NMO” velocities, respectively, used to parametrize a generic TTI medium. Also, the operators \( H_1 \) and \( H_2 \) depend on medium parameters \( \theta \) and \( \phi \), describing the symmetry axis direction.

Assuming the symmetry axis tilt, \( \theta \), to be small and independent of position (constant), and focusing on the 2D case, \( \phi = 0 \) (x-axis). I consider the following trial solutions to equations 4 and 5:

\[
p(x, z, t) \approx p_0(x, z, t) + p_1(x, z, t) \sin \theta + p_2(x, z, t) \sin^2 \theta,
\]

and
Acoustic anisotropic wavefields

\[ q(x, z, t) \approx q_0(x, z, t) + q_1(x, z, t)\sin \theta + q_2(x, z, t)\sin^2\theta. \]  

(7)

Setting \( \theta = 0 \) in equations 4 and 5 allows us to obtain the acoustic wave equation for VTI media, as follows:

\[ \frac{\partial^2 p_0}{\partial t^2} - \nu_H^2 H'_L[p_0] - \nu_v^2 H'_T[q_0] = 0, \]  

(8)

\[ \frac{\partial^2 q_0}{\partial t^2} - \nu_v^2 H'_L[q_0] - \nu_v^2 H'_T[p_0] = 0, \]  

(9)

where \( H'_L \) and \( H'_T \) are the VTI differential operators, and as described previously \( H'_L = H_1(\theta = 0) = \partial^2 / \partial z^2 \) and \( H'_T = H_2(\theta = 0) = \partial^2 / \partial x^2 \). This will allow me to solve for \( p_0 \) and \( q_0 \).

If I insert the trial solutions 6 and 7, respectively, into equations 4 and 5, and I expand the \( \cos \theta = \sqrt{1 - \sin^2 \theta} \) using Taylor’s series for small \( \sin \theta \), I can rearrange the terms for both equations to have the general form \( \sum_i a_i \sin^i(\theta) = 0 \). For this polynomial to equal zero for a nonzero \( \theta \), the coefficients \( a_i \) must equal zero, for \( i = 0, 1, 2, \ldots \). Thus, \( a_0 = 0 \) results in equations 8 and 9. While \( a_1 = 0 \) allows me to first-order coefficients \( p_1 \) and \( q_1 \) to satisfy a similar VTI wave equation, but with a source function that depends on the background VTI wavefields (\( p_0 \) and \( q_0 \)), as follows:

\[ \frac{\partial^2 p_1}{\partial t^2} - \nu_H^2 H'_L[p_1] - \nu_v^2 H'_T[q_1] = 2\nu_H^2 H^c \zeta[p_0] - 2\nu_v^2 H^c \zeta[q_0], \]  

(10)

\[ \frac{\partial^2 q_1}{\partial t^2} - \nu_v^2 H'_L[q_1] - \nu_v^2 H'_T[p_0] = 2\nu_v^2 H^c \zeta[q_0] - 2\nu_v^2 H^c \zeta[p_0], \]  

(11)

where \( H^c = \frac{\partial}{\partial v_L} \). The second-order coefficients \( p_2 \) and \( q_2 \) also satisfy the same VTI wave equation, extracted from \( a_2 = 0 \), as follows:

\[ \frac{\partial^2 p_2}{\partial t^2} - \nu_H^2 H'_L[p_2] - \nu_v^2 H'_T[q_2] = \nu_H^2 [2H^c \zeta[p_1] + H'_L[p_0] - H'_V[p_0]] - \nu_v^2 [2H^c \zeta[q_1] + H'_T[q_0] - H'_V[q_0]], \]  

(12)

\[ \frac{\partial^2 q_2}{\partial t^2} - \nu_v^2 H'_L[q_2] - \nu_v^2 H'_T[p_2] = \nu_v^2 [2H^c \zeta[q_1] + H'_T[q_0] - H'_V[q_0]], \]  

(13)

As I go to higher order, it is clear that the coefficients will satisfy the same VTI wave equation with more complicated source functions. Thus, accuracy will cost more. For 3D media, I will need to solve equations 4 and 5 for VTI media, with source functions that include additional derivatives in \( y \) that depend on the order of the desired coefficients for the expansion.

**Perturbation in \( \eta \)**

In this case, I consider the following trial solutions to the TTI wave equations 4 and 5:

\[ p(x, z, t) \approx p_0(x, z, t) + p_1(x, z, t)\eta + p_2(x, z, t)\eta^2, \]  

(14)

and

\[ q(x, z, t) \approx q_0(x, z, t) + q_1(x, z, t)\eta + q_2(x, z, t)\eta^2. \]  

(15)

Setting \( \eta = 0 \) in equations 4 and 5 allows me to obtain the acoustic wave equation for tilted elliptical anisotropic media, as follows:

\[ \frac{\partial^2 p_0}{\partial t^2} - \nu_H^2 H_2[p_0] - \nu_v^2 H_1[q_0] = 0, \]  

(16)

\[ \frac{\partial^2 q_0}{\partial t^2} - \nu_v^2 H_2[q_0] - \nu_v^2 H_1[p_0] = 0, \]  

(17)

where \( H_1 \) and \( H_2 \) are the TTI operators described above. Because the two equations are similar, \( q_0 = p_0 \) and satisfy

\[ \frac{\partial^2 p_0}{\partial t^2} - 2\nu_H^2 [H_2[p_0] - v_v^2 H_1[q_0]] = 0. \]  

(18)

The first-order coefficients \( p_1 \) and \( q_1 \) satisfy a similar tilted elliptically anisotropic wave equation, but with a source function that depends on the background elliptically anisotropic wavefield (\( p_0 \) and \( q_0 \)), as follows:

\[ \frac{\partial^2 p_1}{\partial t^2} - 2\nu_H^2 H_2[p_1] - \nu_v^2 H_1[q_1] = 2\nu_H^2 H_2[p_0], \]  

(19)

\[ \frac{\partial^2 q_1}{\partial t^2} - 2\nu_v^2 H_2[q_1] - \nu_v^2 H_1[p_1] = 0. \]  

(20)

The second-order coefficients, \( p_2 \) and \( q_2 \), also satisfy the same TI wave equation as follows:

\[ \frac{\partial^2 p_2}{\partial t^2} - 2\nu_H^2 H_2[p_2] - \nu_v^2 H_1[q_2] = 2\nu_H^2 H_2[p_1], \]  

(21)

\[ \frac{\partial^2 q_2}{\partial t^2} - 2\nu_v^2 H_2[q_2] - \nu_v^2 H_1[q_2] = 0. \]  

(22)

This formulation also allows me to constrain the symmetry axis to be normal to the reflector dip and create a mechanism to invert for \( \eta \) using equations 14 and 15. Again, higher-order coefficients satisfy the same second-order tilted elliptical anisotropic wave equation with a source function dependent on the lower order coefficient solutions. For the \( \eta \) perturbation, the complexity of this source function does not increase as I solve for higher order coefficients.
NUMERICAL EXAMPLES

The accuracy of the perturbation-based solutions depends mainly on the amount of perturbation in the parameters. In the numerical tests, I will solve the TTI wave equation by either perturbing from a background VTI medium, and thus the perturbation parameter is the tilt angle, or perturbing from a background tilted elliptical anisotropic medium, with the perturbation parameter \( \eta \). I compare both perturbation solutions with the exact acoustic solution. The finite-difference approximation for all the wave equations is second order in time and fourth order in space. I start with a homogeneous example.

Homogeneous medium

I consider wave propagation in an acoustic TTI media with velocity equal to \( 2 \text{ km/s}, \delta = 0.05, \) and \( \eta = 0.05 \). The source, given by a Ricker wavelet with a peak frequency of 15 Hz, is set in the middle of the propagation medium. The space sampling interval in \( x \) and \( z \) is 0.008 km. Snapshots of the wavefields at 0.7 s for a tilt angle of 10° are shown in Figure 1. The difference between the exact acoustic solution shown in Figure 1a and the one based on the one-term perturbation expansion (Figure 1b) is generally small (Figure 1c). This observation, of course, depends on the order of the expansion, but more so on the amount of tilt and the strength of anisotropy. Figure 2 shows snapshots of the wavefield obtained from the first-order perturbation for three different tilt angles: 10°, 20°, and 30°. The kinematic errors clearly increase with tilt angle as I compare the snapshots with the exact analytically derived traveltme solution (dashed curve). The difference between these wavefields and the exact acoustic TTI wave equation is shown in Figure 3 for the three tilt angles. The errors increase with the larger perturbation given by the larger tilt angle from vertical. In fact, the magnitude of the error for the 30° tilt is comparable in size to the original wavefield. However, most of these errors, as the kinematics demonstrated, are amplitude related. Also, note the presence of the numerical dispersion in the snapshots of the solution. This is a result of the relatively coarse-spaced grid of 0.008 km used in the finite-difference implementation.

On the other hand, focusing on perturbations in \( \eta \) from an elliptical TTI medium background as an alternative to the tilt perturbation allows me to focus on the \( \eta \) influence. Figure 4 compares the exact TTI solution to that perturbed from a background tilted elliptical anisotropic medium using only the first-order approximation in equations 14 and 15. The difference, shown in Figure 4c, exposes a large angular dependence compared with the errors associated with the tilt perturbation case. This is largely because \( \eta \) has a very strong angular influence on velocity. Figure 5 shows the first-order perturbation solution in \( \eta \) for three different tilts (10°, 20°, and 30°) along with the exact traveltme solution given by the dashed curves. The differences with the exact solution (errors), shown in Figure 6 for the three symmetry direction angles, are very similar in size and do not depend on tilt angle. This is expected because the perturbation is in \( \eta \).

In both cases, I can use the second-order expansion for higher accuracy, but at an increase in computational cost. Figure 7 shows the same difference plots shown in Figure 3, but with the second-order accuracy perturbation. As expected, the differences (errors) are smaller. A similar conclusion is drawn in perturbing from an elliptical anisotropic background, as Figure 8 shows an overall reduction in error compared to that shown in Figure 6.
Inhomogeneous medium

The interface geometry embedded in the Marmousi model (Figure 9) is based, somewhat, on a profile through the Cuanza basin (Versteeg, 1993). The model contains many reflectors, steep dips, and strong velocity variations in the lateral and the vertical directions (with a minimum velocity of 1500 m/s and a maximum of 5500 m/s). The point source considered here is a Ricker wavelet with a 25-Hz peak frequency. I initially consider a constant $\eta$ of 0.05.

![Figure 3](image1.png)

Figure 3. A snapshot of the wavefields' difference between that computed using the first-order perturbation in tilt angle and the exact acoustic solution for a tilt angle of (a) 10°, (b) 20°, and (c) 30° for a medium with $v = 2$ km/s, $\eta = 0.05$, and $\delta = 0.05$.

![Figure 4](image2.png)

Figure 4. A snapshot of the wavefields at time 0.7 s for a source in the center (a) computed using the actual acoustic TTI wave equation, (b) computed using the first-order perturbation in $\eta$, and (c) the difference for a medium with $v = 2$ km/s, $\eta = 0.05$, $\delta = 0.05$, and tilt angle of 10°.

![Figure 5](image3.png)

Figure 5. A snapshot of the wavefields at time 0.7 s for a source in the center computed using the first-order perturbation in $\eta$ for a tilt angle of (a) 10°, (b) 20°, and (c) 30° for a medium with $v = 2$ km/s, $\eta = 0.05$, $\delta = 0.05$. The dashed curves correspond to the exact analytically derived travelttime solution.

![Figure 6](image4.png)

Figure 6. A snapshot of the wavefield difference between that computed using the first-order perturbation in $\eta$ and the exact acoustic solution for a tilt angle of (a) 10°, (b) 20°, and (c) 30° for a medium with $r = 2$ km/s, $\eta = 0.05$, and $\delta = 0.05$. 

Acoustic anisotropic wavefields
Again, using the fourth-order finite-difference approximation in space and second order in time, I solve the wave equation for a source located at the surface at lateral position 5 km. The space sampling interval in \( x \) and \( z \) is 0.012 km. A snapshot of the wavefields at 1.2 s for a tilt angle of 10° is shown in Figure 10. The difference between the exact acoustic solution shown in Figure 10a and the one based on one term of the perturbation expansion in tilt angle (Figure 10b) is generally small (Figure 10c). This observation, of course, depends on the order of the expansion, but more so on the amount of tilt and the strength of anisotropy. Figure 11 shows snapshots of the wavefield obtained from the first-order perturbation in tilt angle for three different tilt angles: 10°, 20°, and 30°. The difference between these wavefields and the exact acoustic TTI wave equation is shown in Figure 12 for the three tilt angles. The errors increase with the larger perturbation given by the larger tilt.

On the other hand, focusing on perturbations in \( \eta \), Figure 13 compares the exact TTI solution to that perturbed from a background tilted elliptical anisotropic medium using only the first-order approximation in equations 14 and 15. The difference, shown in Figure 13c, exposes a large angular dependency compared to the larger tilt. This is largely because \( \eta \) has a very strong angular influence on velocity. Figure 14 shows the first-order perturbation solution in \( \eta \) for three different tilts (10°, 20°, and 30°). The differences with the exact solution (errors) shown in Figure 22 for the three symmetry direction angles are very similar and do not depend on tilt angle. Figure 15 shows the wavefield difference between that computed using the first-order perturbation in \( \eta \) and the exact acoustic solution for a tilt angle of (a) 10°, (b) 20°, and (c) 30° for a medium with \( \eta = 0.05 \) and \( \delta = 0.05 \).

In both cases, the second-order expansion can be used for higher accuracy, but at an increase in the computational cost. Figure 16 shows the same difference plots shown in Figure 12, but with the second-order accuracy perturbation. As expected, the difference (errors) are smaller. A similar conclusion is drawn in perturbing from an elliptical anisotropic background, as Figure 17 shows an overall reduction in error compared to that shown in Figure 22. Again, all images are displayed at the same scale. The elliptically based perturbation has a more angular dependent error than the tilt angle perturbation.

**Complex \( \eta \) spatial distribution**

The approximate solution based on a perturbation in \( \eta \) requires theoretically that \( \eta \) be constant. Likewise, the solution corresponding to perturbations in the symmetry axis angle requires that the symmetry angle be constant. In both cases, smooth variations are tolerable, especially if such smooth variations inject an almost stationary behavior within the seismic predominant wavelength. However, a perturbation in \( \eta \) does not set any restrictions on the complexity of the symmetry axis field, and similarly a perturbation in symmetry axis does not require any restrictions on the \( \eta \) field. Alkhalifah (2000) developed a complex \( \eta \) field for the Marmousi model. Figure 18 shows this \( \eta \) with values ranging between 0 and 0.27. I use this model without smoothing to test the perturbation in the angle of symmetry.

The model parameters as well as the frequency of the source are similar to that used in the previous subsection. The only difference is the complex \( \eta \) model (Figure 18). Thus, Figure 19 shows that for a 10° symmetry axis tilt, the errors for the first-order perturbation are comparable to those experienced earlier. As the tilt increases, the errors increase (see Figure 20). Nevertheless, these errors are reasonable for inversion applications because we may want to search for the best tilt that fits the data.
To test how much smoothness is needed in the complex $\eta$ model in Figure 18 for it to provide acceptable approximate solutions (artifact free) for the perturbations in $\eta$, I convolve this complex $\eta$ model with two different-length pyramid-shaped smoothing operators (triangle along each axis), as shown in Figure 21. This is equivalent to multiplying the $\eta$ model in the Fourier domain by a 2D squared sinc function, which acts as a low-pass filter. Figure 22 shows the errors in the first-order perturbation in $\eta$ solutions for the

Figure 9. The Marmousi velocity model.

Figure 10. A snapshot of the wavefields for a source in the center at time 1.2 s (a) computed using the actual acoustic TTI wave equation, (b) computed using the first-order perturbation in tilt angle, and (c) the difference for a medium with $\eta = 0.05$, $\delta = 0.05$, and tilt angle of 10°.

Figure 11. A snapshot of the wavefields for a source in the center at time 1.2 s computed using the first-order perturbation in tilt angle for a tilt angle of (a) 10°, (b) 20°, and (c) 30° for a medium with $\eta = 0.05$ and $\delta = 0.05$.

Figure 12. A snapshot of the wavefield difference between that computed using the first-order perturbation in tilt angle and the exact acoustic solution for a tilt angle of (a) 10°, (b) 20°, and (c) 30° for a medium with $\eta = 0.05$ and $\delta = 0.05$.

Figure 13. A snapshot of the wavefields at time 1.2 s for a source in the center (a) computed using the actual acoustic TTI wave equation, (b) computed using the first-order perturbation in $\eta$, and (c) the difference for a medium with $\eta = 0.05$, $\delta = 0.05$, and tilt angle of 10°.
two smooth \( \eta \) models using a smoothing operator with a length of 3 samples (Figure 22b), 10 samples (Figure 22a), compared to no smoothing (Figure 22c). Because the difference is with respect to the exact acoustic TTI operator with the same smoothed models, all three difference plots show practically the same magnitude of error. However, for the model with no smoothing applied to \( \eta \) (Figure 22a), there are slightly more errors and possibly some artifacts, specifically around lateral location 4 km and depth around 2 km, as well as in other places. Such artifacts are far less observable as I smooth the \( \eta \) model (recall that I am perturbing in \( \eta \)), and ultimately no potential artifacts are expected for a constant \( \eta \) model.

Figure 14. A snapshot of the wavefields at time 1.2 s for a source in the center computed using the first-order perturbation in \( \eta \) for a tilt angle of (a) 10°, (b) 20°, and (c) 30° for a medium with \( \eta = 0.05 \) and \( \delta = 0.05 \).

Figure 15. A snapshot of the wavefield difference between that computed using the first-order perturbation in \( \eta \) and the exact acoustic solution for a tilt angle of (a) 10°, (b) 20°, and (c) 30° for a medium with \( \eta = 0.05 \) and \( \delta = 0.05 \).

Figure 16. A snapshot of the wavefield difference between that computed using the second-order perturbation in tilt angle and the exact acoustic solution for a tilt angle of (a) 10°, (b) 20°, and (c) 30° for a medium with \( \eta = 0.05 \) and \( \delta = 0.05 \).

Figure 17. A snapshot of the wavefield difference between that computed using the second-order perturbation in \( \eta \) and the exact acoustic solution for a tilt angle of (a) 10°, (b) 20°, and (c) 30° for a medium with \( \eta = 0.05 \) and \( \delta = 0.05 \).
DISCUSSION

Wavefield perturbation allows us to approximately isolate the parts of the wavefield attributable to the perturbed parameters. In this case, the perturbed parameters are the anisotropic parameters and specifically $\eta$ and the symmetry axis direction. Both parameters tend to be relatively small, but they are usually difficult to estimate using conventional methods. Using perturbation theory, I use the wavefield in its full inhomogeneous capable glory to relate changes in it to changes in these medium parameters. It has a Born approximation framework, which is useful for waveform inversion. This framework provides an efficient way to update the wavefield to reflect the anisotropic influence; however, because of the nature of the perturbation, it requires that the perturbation parameters be constant at least over the maximum wavelength. It is also incapable of dealing with large perturbations in medium parameters — specifically, those inducing wavefield perturbations beyond the capability of the polynomial expansion in representing the sinusoidal nature of the wavefield. The length associated with such a limitation is governed by the highest frequency.

The perturbation of the acoustic wavefield, to take into account the anisotropy influence of the medium, allows for avoidance of many of the limitations associated with obtaining such wavefields.

Figure 18. The $\eta$ field for the Marmousi model.

Figure 19. A snapshot of the wavefields at time 1.2 s for a source in the center (a) computed using the actual acoustic TTI wave equation, (b) computed using the first-order perturbation in tilt angle, and (c) the difference for a medium with $\delta = 0.0$, tilt angle of $10^\circ$, and $\eta$ given by Figure 18.

Figure 20. A snapshot of the wavefield difference between that computed using the first-order perturbation in tilt angle and the exact acoustic solution for a tilt angle of (a) $10^\circ$, (b) $20^\circ$, and (c) $30^\circ$ for a medium $\delta = 0.0$, and $\eta$ given by Figure 18.

Figure 21. (a) The $\eta$ model in Figure 18, (b) smoothed using a triangulated filter with length of 0.075 km in each direction, and (c) 0.25 km in each direction.
in the conventional way. The $\eta$ perturbation of the wavefield from a background elliptical anisotropic (vertical or tilted) medium are free of shear-wave artifacts. Unlike the original equation, it also works for negative $\eta$. The perturbation with respect to the tilt angle allows us to rely on a relatively efficient (far more efficient than the TTI) VTI engine for calculating the background solutions and the perturbation coefficients.

The accuracy of the wavefield perturbation solution also depends on the frequency content because lower frequencies yield lower perturbation errors, especially because the perturbation, unlike the classical Born type, is based on a complex background and smooth variation in the perturbed parameter. Thus, it provides a platform for parameter estimation in line with the needs associated with full waveform inversion governed by the phase component of the wavefield. The perturbation in tilt angle yields an overall uniform distribution of error with angle mostly in amplitude. On the other hand, a perturbation in $\eta$ has larger errors in the direction normal to tilt angle, where the influence of $\eta$ is the largest.

**CONCLUSIONS**

Using perturbation theory, I derive partial differential equations to solve for the expansion coefficients of the acoustic anisotropic wavefield solution in terms of $\eta$ and the symmetry axis. Such coefficients provide insights into the sensitivity of the acoustic wavefield to such anisotropy parameters. They also provide solutions free of some of the limitations associated with the original TTI acoustic wave equation. Though the synthetic tests on homogenous and inhomogeneous media showed higher accuracy for the $\eta$ perturbation, the perturbation in $\eta$ and symmetry axis tilt provided generally accurate wavefields.

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**REFERENCES**


