GEOPHYSICS, VOL. 77, NO. 3 (MAY-JUNE 2012); P. U31–U37, 6 FIGS.
10.1190/GEO2011-0465.1

Prestack traveltime approximations

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ABSTRACT

Many of the explicit prestack traveltime relations used in practice are based on homogeneous (or semi-homogenous, possibly effective) media approximations. This includes the multifocusing, based on the double square-root (DSR) equation, and the common reflection stack (CRS) approaches. Using the DSR equation, I constructed the associated eikonal form in the general source-receiver domain. Like its wave-equation counterpart, it suffers from a critical singularity for horizontally traveling waves. As a result, I recasted the eikonal in terms of the reflection angle, and thus, derived expansion based solutions of this eikonal in terms of the difference between the source and receiver velocities in a generally inhomogenous background medium. The zero-order term solution, corresponding to ignoring the lateral velocity variation in estimating the prestack part, is free of singularities and can be used to estimate traveltimes for small to moderate offsets (or reflection angles) in a generally inhomogeneous medium. The higher-order terms include limitations for horizontally traveling waves, however, we can readily enforce stability constraints to avoid such singularities. In fact, another expansion over reflection angle can help us avoid these singularities by requiring the source and receiver velocities to be different. On the other hand, expansions in terms of reflection angles result in singularity free equations. For a homogenous background medium, as a test, the solutions are reasonably accurate to large reflection and dip angles. A Marmousi example demonstrated the usefulness and versatility of the formulation.

INTRODUCTION

Traveltimes or the high-frequency geometrical representation of waves have contributed an abundance of information to our understanding of wave propagation. They are practically the main, or even lone, source of smooth velocity information used typically for processing and imaging or as an initial velocity model for the more elusive waveform-based velocity inversion. Traveltimes corresponding to a wave emanating from a source can be derived directly from the eikonal equation (Vidale, 1990). The conventional eikonal is, however, a one-way traveltime map that does not reflect directly the source/receiver setup associated with our typical surface seismic reflection experiment. To include midpoint and offset, more exotic formulas are available including the double-square-root (DSR), and its direct imaging implementation through the multifocusing method, and the common reflection stack (CRS) (Claerbout, 1995; Cruz et al., 2000; Jager et al., 2001; Gurevich and Landa, 2002). In fact, Alkhalifah (2000) use a modified DSR formulation to develop the traveltime offset-midpoint relation for transversely isotropic media. However, all these relations are based on a general homogenous medium assumption, but can handle a smoothly varying medium under effective medium theory (i.e., Dix, 1955; Backus, 2000).

The DSR relation describes the traveltime field for a noncoincident source and receiver given along a flat surface, and its Fourier form relates the vertical slowness to the horizontal one in what is commonly referred to as the dispersion relation. It is typically used to perform downward continuation through survey-sinking (Claerbout, 1985). Recently, Iversen (2004) and Duchkov and de Hoop (2009) use the DSR formula to extrapolate isochron rays to help map prestack images. These rays represent the characteristics of a new Hamiltonian based on the DSR formula. In addition to the inherent singularity in the DSR formulation for horizontally traveling rays, these rays suffer from all the limitations commonly associated with ray methods including the requirement for a smooth velocity model and the unease in mapping the traveltime onto a regular grid. Some of these limitations can be alleviated using an Eulerian implementation.

Direct finite-difference solutions of the DSR-based eikonal will evidently suffer from the essential singularity inherent in the DSR formulation. Unlike ray methods, it is hard to isolate (limit the angles) and avoid the singularity. As a result, I use the power of

Manuscript received by the Editor 24 November 2011; revised manuscript received 20 January 2012; published online 5 April 2012.
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perturbation theory to formulate an Eurlerian implementation that avoids this singularity. This can be accomplished by realizing that the zero-offset conventional eikonal does not suffer from such a problem. Thus, perturbing from a zero-offset model setting should allow us to approximately estimate the prestack traveltime in a generally background inhomogeneous medium for, at least, small offsets.

In this paper, I present practical and useful approximations to the DSR-based Hamiltonian that does not suffer from the essential singularity in the conventional DSR formula. This perturbation-based eikonal provides a mechanism to instantaneously predict approximate traveltimes for nonzero-offset settings, which has a direct application in velocity estimation. These approximations provide insights into the angular and reflector-dip dependency considering an inhomogeneous medium background. The accuracy of these approximations is tested in the homogenous background case. I also show that prestack traveltime for the Marmousi model can be readily estimated.

THEORY

Consider a seismic survey of sensors recording a wavefield at a surface at depth z as a function of time t and source and receiver locations $\mathbf{s}$ and $\mathbf{r}$. In the geometrical (high-frequency) approximation, the phase function, represented by the traveltime $\tau(\mathbf{s}, \mathbf{r}, z)$ satisfies the appropriate dispersion relation, which is, in the case of prestack data, the DSR equation (Claerbout, 1985) given by

$$\frac{\partial \tau}{\partial z} = \frac{1}{2}\left(\sqrt{\frac{1}{v^2(s,z)} - |\nabla_s \tau|^2} + \sqrt{\frac{1}{v^2(r,z)} - |\nabla_r \tau|^2}\right), \quad (1)$$

where $v(x,z)$ is the velocity of the medium, and $(\mathbf{s}, \mathbf{r})$ reduces to (s, r) for the 2D case.

To derive a partial differential equation (PDE) representing the behavior of the wavefront singularities in the extended source-receiver domain, I cast the DSR relation 1 in a polynomial form in terms of the slownesses and consider the 2D case as follows:

$$\begin{aligned} &v_-^4 + 4v_s^2 v_r^2 [v_-^2 (p_s^2 - p_r^2) - 4v_+^2 p_z^2] \\ &\quad + 4v_s^4 v_r^4 [(p_s^2 + p_r^2 + 4p_z^2)^2 - 4p_r^2 p_s^2] = 0, \end{aligned} \quad (2)$$

where $p_s = \frac{\partial \tau}{\partial s}$, $p_r = \frac{\partial \tau}{\partial r}$, $p_z = \frac{\partial \tau}{\partial z}$ and $2v_+^2 = v_s^2 + v_r^2$, $2v_-^2 = v_s^2 - v_r^2$. Recall, v_s and v_r are the source and receiver velocities, respectively, and can be represented as $v(s,z)$ and $v(r,z)$, because the velocity for the 2D medium is a 2D function regardless of the extension of the domain to source and receiver axes. Equation 2 represents the Hamiltonian form for the propagation of singularities of the prestack wavefield. However, this form suffers from two critical problems. In the case of $v(z)$ media, $v_- = 0$, and thus, the form (order) of the Hamiltonian changes dramatically, reflecting a singularity. Another more critical singularity, even for the case of $v(z)$ media, is given by $p_z = 0$, which corresponds to waves traveling horizontally. Solving equation 2 using its characteristics (Duchkov and de Hoop, 2009), allows us to control the angles of the rays and avoid horizontally traveling waves, or $p_z = 0$. A conventional Eurlarian implementation of equation 2 does not provide us with such luxury. More exotic forms based on a space-phase representation like the Liouville form might be required here (Duistermaat, 1974).

USING PERTURBATION THEORY

The Hamiltonian in equation 2 reduces to our familiar eikonal for waves emanating from a source when we set $p_s = p_r$ and $v = v_s = v_r$. Specifically, defining the half-offset slowness, $p_h = p_r - p_s = \frac{\partial \tau}{\partial h}$ and the midpoint slowness $p_x = p_r + p_s = \frac{\partial \tau}{\partial x}$, equation 2 reduces to $p_x^2 + p_z^2 = \frac{1}{v^2}$ when $p_h = 0$ and $v_- = 0$, which is free of singularities. Naturally, to construct a stable solution, we will need to perturb from this stable zero-offset solution.

Considering that a convenient and useful representation of the offset axis is given by the scattering (reflection) angle, I transform the half-offset slowness to reflection angle using $\tan\theta = \frac{p_h}{p_z}$. In addition, by substituting $v_s^2 = v_+^2 - v_-^2$ and $v_r^2 = v_+^2 + v_-^2$, we end up with the following eikonal:

$$\begin{aligned} &v_-^4 - 4(v_+^4 - v_-^4)[-v_-^2 p_x p_z \tan(\theta) + v + p_z^2] \\ &\quad + (v_+^4 - v_-^4)^2[(p_x^2 + p_z^2 \tan(\theta) + 2p_z^2)^2 \\ &\quad - (p_x^2 - p_z^2 \tan(\theta))^2] = 0. \end{aligned} \quad (3)$$

Taking the difference between the source and receiver velocities, v_-, to be small and independent, and because equation 3 has only even powers in v_-, I consider the following trial solution

$$\tau(x,z,\theta) \approx \tau_0(x,z) + \tau_v(x,z)v_-^2 + \tau_{v_2}(x,z)v_-^4. \quad (4)$$

Inserting equation 4 into equation 3, with the slownesses replaced by their respective traveltime derivatives, and setting $v_- = 0$, yields the following eikonal:

$$v_+^2 \left(\frac{\partial \tau_0}{\partial x}\right)^2 + v_+^2 \left(\frac{\partial \tau_0}{\partial z}\right)^2 = \cos(\theta^2), \quad (5)$$

which is equivalent to the conventional eikonal with the velocity (squared average of the source and receiver velocity) scaled by the cosine of the reflection angle. This factor is very similar to what we observe for the dip moveout correction (Levin, 1985). We can easily solve this equation conventionally by weighting the interval velocity by the cosine of reflection angle. It represents the exact eikonal (high-frequency asymptotic version) for prestack traveltimes for a reflection angle θ in homogeneous media, granted θ is considered constant. In inhomogeneous media, it is an approximation that clearly depends on the amount of lateral velocity variation over offset, but it is also exact for $\theta = 0$ (zero offset).

The term corresponding to the v_-^2 coefficient of our trial solution is a linear first-order partial differential equation given by

$$2\frac{\partial \tau_0}{\partial x}\frac{\partial \tau_v}{\partial x} + 2\frac{\partial \tau_0}{\partial z}\frac{\partial \tau_v}{\partial z} = -v_+^2 \sin(\theta)\cos(\theta)\frac{\frac{\partial \tau_0}{\partial x}}{\frac{\partial \tau_0}{\partial z}}. \quad (6)$$

The factor at the end of the right-hand side of this equation is nothing but the effective reflector dip in the background medium, with $\tan\phi_0 = \frac{\frac{\partial \tau_0}{\partial x}}{\frac{\partial \tau_0}{\partial z}}$, ϕ_0 corresponds to the reflector dip. For a vertical reflector this term tends to infinity, which is a problem. Also, for a horizontal reflector $\tan\phi_0 = 0$, and thus, $\tau_v = 0$, which implies that the second-order term does not contribute to the traveltime solution (equation 4) for a horizontal reflector in the background medium.

This might be expected, because for horizontal reflectors, the influence of the linear portion of the velocity variation laterally tend to cancel out over the full offset.

The term corresponding to the v_-^4 coefficient is also linear first order, given by

$$\begin{aligned} & 8\sec(\theta^2)\, v_+^8 \frac{\partial\tau_0}{\partial x}\left(\frac{\partial\tau_0}{\partial z}\right)^2 \frac{\partial\tau_{v_2}}{\partial x} + 8\sec(\theta^2)\, v_+^8 \left(\frac{\partial\tau_0}{\partial z}\right)^3 \frac{\partial\tau_{v_2}}{\partial z} \\ & = -4v_+^2\left(\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_v}{\partial x}\frac{\partial\tau_0}{\partial z}\, v_+^2\left(4\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_v}{\partial z}\, v_+^4\sec^2(\theta) + \tan(\theta)\right)\right. \\ & + \frac{\partial\tau_0}{\partial x}\frac{\partial\tau_v}{\partial z}\, v_+^2 \tan(\theta) \\ & \left. + \left(\frac{\partial\tau_0}{\partial z}\right)^2\left(v_+^6\left(\left(\frac{\partial\tau_v}{\partial x}\right)^2 + 5\left(\frac{\partial\tau_v}{\partial z}\right)^2\right)\sec^2(\theta) - 1\right)\right) - 1. \end{aligned} \tag{7}$$

For a zero-dip reflector in the background medium, $\frac{\partial\tau}{\partial x} = 0$, and equation 7 reduces to

$$\begin{aligned} & 8\sec(\theta^2)\, v_+^8 \frac{\partial\tau_0}{\partial x}\left(\frac{\partial\tau_0}{\partial z}\right)^2 \frac{\partial\tau_{v_2}}{\partial x} + 8\sec(\theta^2)\, v_+^8 \left(\frac{\partial\tau_0}{\partial z}\right)^3 \frac{\partial\tau_{v_2}}{\partial z} \\ & = -4v_+^2\left(\frac{\partial\tau_0}{\partial z}\right)^2 \\ & \times\left(v_+^6\left(\left(\frac{\partial\tau_v}{\partial x}\right)^2 + 5\left(\frac{\partial\tau_v}{\partial z}\right)^2\right)\sec^2(\theta) - 1\right) - 1. \end{aligned} \tag{8}$$

In this case, τ_{v_2} is not necessarily zero, and thus, we might have contributions from this higher-order term of the traveltime solution (equation 4 for a horizontal reflector. This equation also is singular for $\frac{\partial\tau_0}{\partial z} = 0$. The singularities in equations 6 and 7 can be handled because they correspond to the background media solution.

These linear partial differential equations must be solved in succession starting with τ_v. As soon as the τ_v coefficient is evaluated, they can be used, as Alkhalifah (2011) shows, to estimate the traveltime using the first sequence of Shanks transform (Bender and Orszag, 1978), which has the form

$$\tau(x,z,\theta) \approx \tau_0(x,z) + \frac{v_-^2\tau_v^2(x,z)}{\tau_v(x,z) - v_-^2\tau_{v_2}(x,z)}. \tag{9}$$

Equation 9 provides the traveltime formula in the plane wave domain, as the reflection angle corresponds to plane waves. Specifically, the traveltime in equation 9, as well as those in the previous equations correspond to a conventional τ-p transform of the space version. A simple (τ-p)-like inverse transform provides the space domain equivalent, angle moveout, as follows:

$$\begin{aligned} t(x,z,\theta) &\approx \tau_0(x,z,\theta) + \frac{v_-^2\tau_v^2(x,z,\theta)}{\tau_v(x,z,\theta) - v_-^2\tau_{v_2}(x,z,\theta)} \\ &+ \frac{z\,\sin(\theta)\,\tan(\theta)}{v_+(x,z)}, \end{aligned} \tag{10}$$

because $p_h = \frac{\sin(\theta)}{v_+(x,z)}$.

To solve the above formulations, we will have to know $\theta(x,z)$, granted that θ at the image point is given by the desired reflection angle. We can always consider θ to be constant, which is true for zero offset, but a good approximation for small offsets, and exact again for homogeneous media. Although v_+ and v_- are not tractable (cannot be evaluated because we are missing the subsurface offset information) under this perturbation approach, having the freedom to insert lateral velocity variation should be beneficial. It is clearly a use in estimating v_+ and v_- from a background medium based on $v_+ = v$ and $v_- = 0$.

A HOMOGENEOUS MODEL TEST

Although the above equations are developed for a general inhomogeneous background medium, I examine their accuracy in representing traveltime moveout in the homogeneous case. This is convenient because the solutions for the homogeneous case are readily available for comparison.

I use the simple traveltime relation for a homogeneous background to recursively solve for the coefficients of the traveltime expansion in v_-, and thus, obtain analytical representations for coefficients τ_0, τ_v, and τ_{v_2}. For a homogeneous background medium, a solution to equation 5 is given analytically as follows

$$\tau_0(x,z,\theta) = \cos(\theta)\sqrt{\frac{x^2+z^2}{v_+^2}}. \tag{11}$$

Using equations 6 and 7 in succession, I obtain

$$\tau_v(x,z,\theta) = \frac{x\,\sin(\theta)\sqrt{\frac{x^2+z^2}{v_+^2}}}{2v_+^2 z}, \tag{12}$$

and

$$\tau_{v_2}(x,z,\theta) = -\frac{\sec(\theta)\sqrt{\frac{x^2+z^2}{v_+^2}}(-5x^4 + 9x^2z^2 + (5x^4 - 7x^2z^2 - 7z^4)\,\cos(2\theta) + z^4)}{16v_+^4 z^4}. \tag{13}$$

We can use the first- or second-order approximations (equation 4) to predict prestack traveltimes. In this case, Shanks transform did not help as it produced solutions similar to the first-order approximation. Figure 1a shows the traveltime curves corresponding to zero-dip reflector at the image point, where the second-order approximation shown by dashed gray curve is closer to the analytical solution (solid curve) than the first-order form shown by the dashed black blue curve. The exact solution is derived directly from the DSR formula through considering v_s and v_r independent parameters. This observation holds to a lesser degree for a dipping reflector given by $x = 0.5$ km in Figure 1b, and errors are generally higher as the reflection angle range becomes smaller for larger dip angles.

EXPANDING IN REFLECTION ANGLE

For small offsets, the reflection angle tends to be small. In Appendix A, I present the traveltime solutions for the general case of expanding in θ for v_- not necessarily equal to zero. However, for simplicity, setting v_- to equal zero, and thus, ignoring the change of velocity laterally for the reflection angle expansion will allow us to focus on the influence of reflection angle on traveltimes. Taking the scattering angle θ to be small and independent, I consider the following trial solution:

$$\tau(x,z,\theta) \approx \tau_0(x,z) + \tau_\theta(x,z)\tan^2(\theta) \quad + \tau_{\theta_2}(x,z)\tan^4(\theta). \tag{14}$$

Because traveltime is symmetric with respect to the scattering angle (Snell's law and the principal of reciprocity), the proposed solution has only even powers of tan θ. Substituting the trial solution 14 into the eikonal 3 and setting all variables corresponding to the expansion terms to zero yields a set of PDEs starting with the conventional eikonal corresponding to the zeroth-order term

$$v_+^2\left(\left(\frac{\partial\tau_0}{\partial x}\right)^2 + \left(\frac{\partial\tau_0}{\partial z}\right)^2\right) = 1. \tag{15}$$

The other equations corresponding to the higher-order terms are first-order linear PDEs having the following general form (See Appendix A):

$$v_+^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_i}{\partial x} + v_+^2\frac{\partial\tau_0}{\partial z}\frac{\partial\tau_i}{\partial z} = f_i(x,z), \tag{16}$$

with $i=\theta,\theta_2$, and $i=0$ satisfies the eikonal equation for a zero-offset background model, where $f_0=1$. The function $f_1=-\frac{1}{2}$ and f_2 includes terms corresponding to the $\tan\theta$ coefficient (see Appendix A). Therefore, these linear partial differential equations also must be solved in succession starting with $i=\theta$. Again, as soon as the τ_θ coefficient is evaluated, to estimate the traveltime using the first-sequence of Shanks transform (Bender and Orszag, 1978), which has the form

$$\tau(x,z,\theta) \approx \tau_0(x,z) + \tau_v(x,z)v_-^4 + \frac{\tan^2(\theta)\tau_\theta^2(x,z)}{\tau_\theta(x,z) - \tan^2(\theta)\tau_{\theta_2}(x,z)}. \tag{17}$$

A HOMOGENEOUS MODEL TEST FOR THE θ EXPANSION

Again, I use the simple traveltime relation for a homogeneous background to recursively solve for the coefficients of the traveltime expansion in θ, and thus, obtain analytical representations for coefficients τ_0, τ_θ, and τ_{θ_2}. Setting $v_-=0$, and $v=v_+$, to allow for a simplified presentation, I obtain, using expansion 14, an analytical representation of traveltime for nonzero θ given by

$$\tau(x,z,\theta) = \frac{1}{8}(8 - 4\tan^2(\theta) + 3\tan^4(\theta))\sqrt{\frac{x^2+z^2}{v^2}}. \tag{18}$$

Using Shanks transform, we can modify the second-order form 18 to an effectively infinite series in θ given by

$$\tau(x,z,\theta) = \frac{(5+3\cos(2\theta))\sqrt{\frac{x^2+z^2}{v^2}}}{7+\cos(2\theta)}. \tag{19}$$

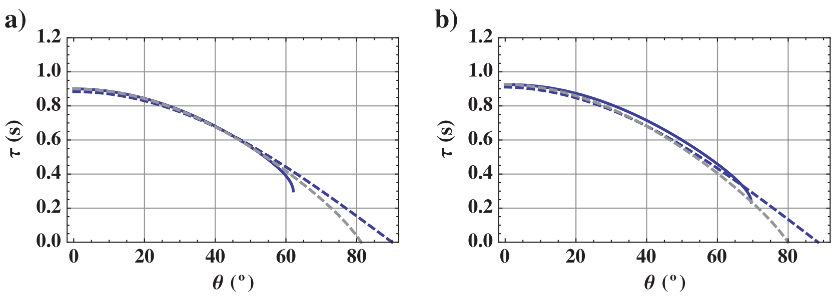

Figure 1. The traveltime as a function of the reflection angle θ for an image point at depth 2 km and (a) lateral offset $x=0$ (zero-dip) and (b) $x=0.5$ km (14° dip). The solid curve corresponds to the exact solution extracted from the DSR formula (considering v_s and v_r independently), the dashed blue curve corresponds to the first order using equation 12, and the dashed gray curve corresponds to the second order using equation 13. Here, $v_s=2$ km/s and $v_r=2.5$ km/s.

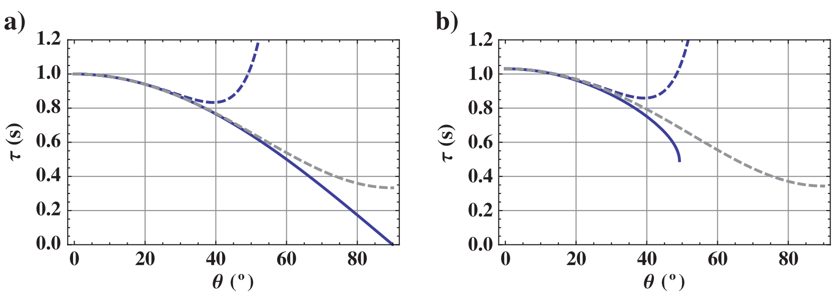

Figure 2. The traveltime as a function of the reflection angle θ for an image point at depth 2 km and (a) lateral offset $x=0$ (zero-dip) and (b) $x=0.5$ km (14° dip). The solid curve corresponds to the exact solution, the dashed blue curve corresponds to the second-order formula 18, and the dashed gray curve corresponds to the Shanks transform 19. Here, the velocity is 2 km/s.

Using a zero-offset homogeneous background model with a velocity equal to 2 km/s, I compare the traveltime of the approximations extracted from the DSR eikonal-based formulations (equations 18 and 19 with the exact formula for this homogenous model, given by the DSR itself. For a reflector at depth $z=2$ km and an image point below the midpoint position, $x=0$ (zero-dip), Figure 2a shows that the Shanks transform based approximation shown by dashed gray curve is closer to the analytical solution (solid curve) than the second-order form shown by the dashed black blue curve. This observation holds even for a dipping reflector given by $x=0.5$ km in Figure 2b, however, with larger errors as the reflection angle range becomes smaller for larger dip angles. In this case, we benefit from the Shanks higher-order transform to improve the expansion approximation accuracy and that is reflected in the results. The plot of the difference in traveltime between the exact solution and the Shanks transform-based approximation as a function of reflection angle and x is shown in Figure 3. The errors for small offsets are generally small (almost zero) for most dips. As the maximum reflection angle decreases with increasing dip, the errors show up at lower reflection angles until the maximum angle tends to zero for a 90° dip reflector.

Although the homogeneous medium versions of the newly developed approximations show high accuracy, the real power of the new formulations is in predicting prestack travelimes in inhomogeneous media.

MARMOUSI TEST

Using the Marmousi model, we can solve for the traveltime emanating from a source using the conventional eikonal. This forms the background traveltime τ_0 in equation 14. The same velocity model can be used to evaluate the other coefficients of the expansion using equation 16. Figure 4 shows traveltime contours (solid) for the zero-offset wavefield for a source located near the assumed reservoir, contours (elongated dashed) for the prestack traveltime with a reflection angle of 5°, and contours (dashed) for the prestack traveltime with a reflection angle of 10°. Clearly, for the prestack case, the traveltime given in the angle domain (plane waves) is slower than its zero-offset counterpart. However, more importantly, we have managed to readily predict the traveltime for the nonzero-offset case for the complex Marmousi model (shown in the background of Figure 4).

One clear objective of these formulations is to instantly calculate traveltime moveout as a function of angle. This will allow for efficient scans for media parameters such as velocity. The traveltime response on the surface for the source in Figure 4 is displayed in Figure 5 for a range of θ values all computed using the Shanks transform analytical formula, equation 17. The coefficients of the expansion are computed using equation 16 for the Marmousi velocity model. The range of scattering angles θ used here is between 0° and 30° and has a reasonably large influence on the traveltime field with the fastest arrivals corresponding to $\theta = 0°$. Note that the smallest traveltime is not above the source at 6700 m, which is a testament to the complexity of the velocity field. Also, interesting is that the angle dependency (distance between curves) increases at large angles (similar to the offset case).

The dependence of the traveltime at a potential image point, given here at the source location of Figure 4, on the scattering angle given by $\tan(\theta)^2$ and dip given by lateral position is illustrated in Figure 6. After obtaining the coefficient of the perturbation numerically, the traveltimes in Figure 6 are obtained analytically using equation 17. This allows for a wide range of traveltime information readily available from a few numerical solutions (in this case, three).

INCLUDING DIP ANGLE

A natural extension to this perturbation implementation is an eikonal representation in dip and reflection angle considering a background $v(z)$ medium. In this case, we define the dip angle ϕ as $\tan\phi = \frac{p_x}{p_z}$ and insert it into the eikonal

$$v_-^4 - 4(v_+^4 - v_-^4)[v_-^2 p_z^2 \tan\theta \tan\phi + v_+^2 p_z^2] + (v_+^4 - v_-^4)^2 \\ p_z^4[(\tan\phi^2 + \tan\theta^2 + 2)^2 - (\tan\phi^2 - \tan\theta^2)^2] = 0. \tag{20}$$

Taking $v_- = 0$, simplifies equation 20 to

$$v_+^2 p_z^2 = \cos\phi \cos\theta. \tag{21}$$

Solving equation 24 yields the one-way traveltime solution

$$\tau_0(z) = \int_0^z \frac{\cos\theta(\xi)\cos\phi(\xi)}{v_+(\xi)} d\xi, \tag{22}$$

where $p_z = \frac{\partial\tau_0}{\partial z}$.

We can again include a factor for correcting for velocity variation laterally between the source and receiver and consider the following trial solution for equation 20:

$$\tau(x, z, \theta) \approx \tau_0(x, z) + \tau_v(x, z) v_-^2 + \tau_{v_2}(x, z) v_-^4, \tag{23}$$

by considering v_- to be small and independent. The zeroth-order solution is given by t_0 in equation 22, whereas τ_v satisfies the following linear first-order equation:

$$v_+^2 \frac{\partial\tau_v}{\partial z} = -\tan(\theta)\tan(\phi)\frac{\partial\tau_0}{\partial z}, \tag{24}$$

with an integral solution given by

$$\tau_v(z) = -\frac{1}{2}\int_0^z \frac{\sin\theta(\xi)\sin\phi(\xi)}{v_+^3(\xi)} d\xi. \tag{25}$$

For zero-dip or zero-reflection angle, the first-order lateral variation does not contribute to the traveltime solution. Finally, τ_{v_2} satisfies the following equation

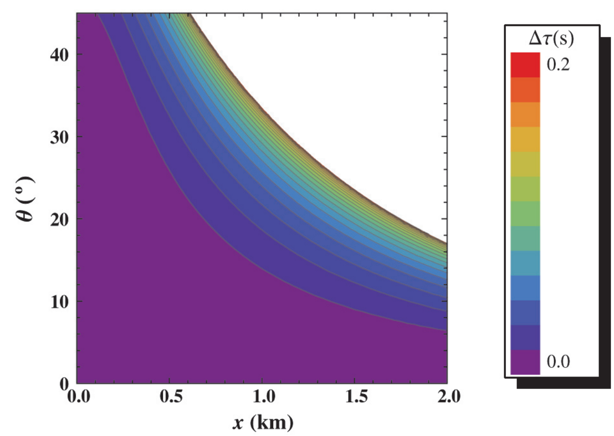

Figure 3. The traveltime difference between the exact (DSR) formulation and Shanks-based approximation as a function of reflection angle and image point lateral offset x.

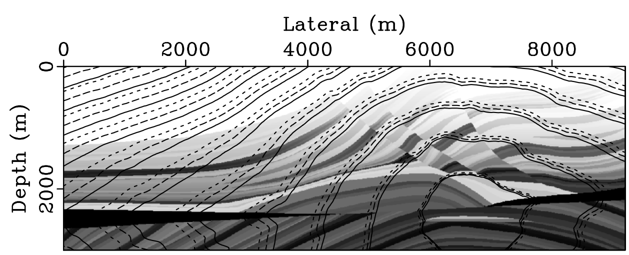

Figure 4. Traveltime contours for the zero-offset case (solid curves), for $\theta = 5°$ (elongated dashed curves), and that for $\theta = 10°$ (dashed curves) using the Shanks transform expansion for a source located at lateral position 6700 m and depth of 2400 m. The Marmousi model is shown in the background.

$$8v_+^6\frac{\partial\tau_0}{\partial z}\frac{\partial\tau_{v_2}}{\partial z}=4v_+^2\left(\frac{\partial\tau_0}{\partial z}\right)^2-4v_+^6\left(\frac{\partial\tau_v}{\partial z}\right)^2-1, \tag{26}$$

with an integral solution given by

$$\tau_{v_2}(z)=\frac{1}{16}\int_0^z\frac{\sec\phi(\xi)(\cos\theta(\xi)(3\cos(2\phi(\xi))+5)+\sec\theta(\xi)(\cos(2\phi(\xi))-3))}{v^5(\xi)}d\xi. \tag{27}$$

In $v(z)$ media, unlike general inhomogeneous media, we can approximate the actual offset needed to determine v_- and v_+ by also computing the offset as a function of depth for a laterally inhomogeneous background medium. Because $p_h=\tan\theta(z)p_z$, the offset is given by

$$X(z)=2\int_0^z\frac{p_h v(\xi)d\xi}{\sqrt{1-v^2(\xi)p_h^2}}, \tag{28}$$

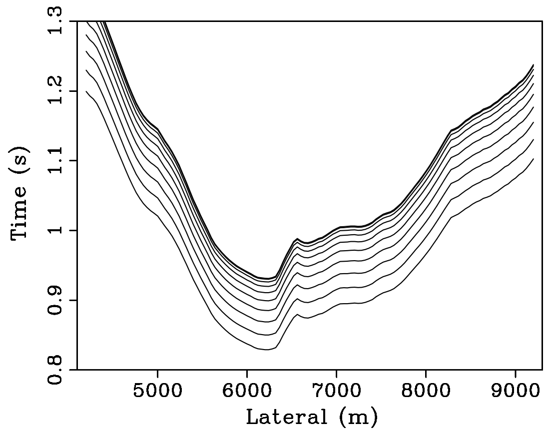

Figure 5. Traveltime as a function of lateral position measured at the surface (depth equal zero) for the same source in Figure 4 for a range of θ between 0° and 30° at 3° increments.

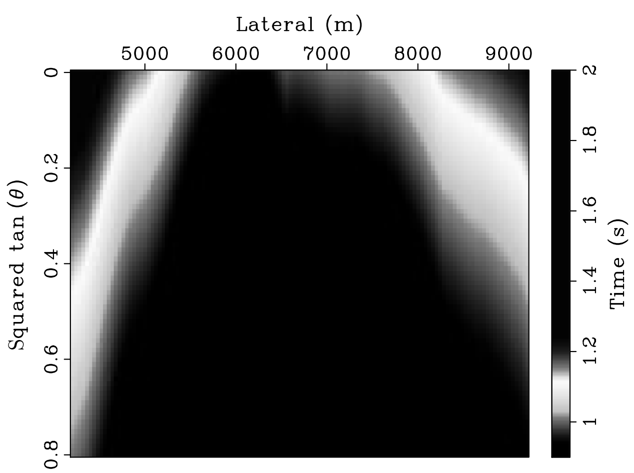

Figure 6. Traveltime as a function of lateral position and $\tan(\theta^2)$ calculated at the surface for the source in Figure 4. It illustrates the traveltime dependency of a potential image point (source) for different dips and scattering angles.

where v is the velocity of the background $v(z)$ medium. In fact, we can also evaluate $\theta(z)$ and $\phi(z)$.

CONCLUSIONS

The geometrical description of the wavefield in the prestack domain offers opportunities for high-frequency imaging and velocity estimation. I developed approximations to the DSR formulation based on a Hamiltonian representation that allows for analytical expansions of the traveltime in a general inhomogeneous background medium. More specifically, I focused on expansions with respect to the difference between the source and receiver velocities and the reflection angle, both of which can be considered small. An application to the Marmousi model shows the versatility of the approach. Additional expansions over dips and reflection angles provided insights into the perturbation features of the DSR formula. It shows that lateral velocity variation tend to not influence the leading-order behavior of traveltimes for horizontal reflectors even in complex media.

ACKNOWLEDGMENTS

I thank Alexey Stovas for many stimulating discussions on this subject. I thank KAUST for its support. I also thank the associate editor and two anonymous reviewers for their helpful suggestions.

APPENDIX A

EXPANSION IN θ

To derive a traveltime equation in terms of perturbations in reflection angle, θ, we first establish the form for the governing equation for prestack media given by the eikonal representation. The prestack eikonal equation 3 for P-waves in isotropic media in 2D (for simplicity) is given by

$$v_-^4-4(v_+^4-v_-^4)\left[v_-^2\frac{\partial\tau}{\partial x}\frac{\partial\tau}{\partial z}\tan\theta+v_+^2\left(\frac{\partial\tau}{\partial z}\right)^2\right]+(v_+^4-v_-^4)^2. \tag{A-1}$$

$$\left[\left(\left(\frac{\partial\tau}{\partial x}\right)^2+\left(\frac{\partial\tau}{\partial z}\right)^2\tan^2(\theta)+2\left(\frac{\partial\tau}{\partial z}\right)^2\right)^2-\left(\left(\frac{\partial\tau}{\partial x}\right)^2-\left(\frac{\partial\tau}{\partial z}\right)^2\tan^2(\theta)\right)^2\right]=0. \tag{A-2}$$

To solve equation A-1 through perturbation theory, we assume that θ is small and independent, I consider the following trial solution

$$\tau(x,z,\theta)\approx\tau_0(x,z)+\tau_\theta(x,z)\tan^2(\theta)\quad+\tau_{\theta_2}(x,z)\tan^4(\theta), \tag{A-3}$$

where τ_0, τ_θ, and τ_{θ_2} are coefficients of the expansion given in units of traveltime, and, for practicality, terminated at the fourth power of $\tan\theta$. Because traveltime is symmetric with respect to the scattering angle (Snell's law and the principal of reciprocity), the proposed solution has only even powers of $\tan\theta$. Substituting the trial solution A-3 into the eikonal 3 and setting all variables corresponding to the expansion terms to zero yields a set of PDEs starting with the conventional eikonal corresponding to the zeroth-order term

$$4(v_-^4 - v_+^4)\left(\frac{\partial\tau_0}{\partial z}\right)^2 \times\left((v_-^4 - v_+^4)\left(\left(\frac{\partial\tau_0}{\partial x}\right)^2 + \left(\frac{\partial\tau_0}{\partial z}\right)^2\right) + v_+^2\right) = -v_-^4. \quad \text{(A-4)}$$

This equation is clearly singular if $v_- = 0$, which is easy to avoid because it is a medium parameter, and then we can set to a small number. We also could expand around $v_- = 0$, as shown in the text. Equation A-4 is more nonlinear than the conventional eikonal. It actually has a form very similar to the anisotropic eikonal (Alkhalifah, 2002). At least this form does not include a singularity based on parameters not easily controllable, like our background solution. The first-order coefficient τ_θ satisfies the following linear PDE equation:

$$\begin{aligned}
&2(v_-^4 - v_+^4)\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial z}\frac{\partial\tau_\theta}{\partial x} \\
&+\left(2\left((v_-^4 - v_+^4)\left(\frac{\partial\tau_0}{\partial x}\right)^2 + v_+^2\right)\right. \\
&\left.+4(v_-^4 - v_+^4)\left(\frac{\partial\tau_0}{\partial z}\right)^2\right)\frac{\partial\tau_\theta}{\partial z} \\
&= -(v_-^4 - v_+^4)\left(\frac{\partial\tau_0}{\partial x}\right)^2\frac{\partial\tau_0}{\partial z} - (v_-^4 - v_+^4)\left(\frac{\partial\tau_0}{\partial z}\right)^3. \quad \text{(A-5)}
\end{aligned}$$

The second-order coefficient τ_{θ_2} also satisfies a linear PDE equation given by

$$\begin{aligned}
&2(v_-^4 - v_+^4)\frac{\partial\tau_0}{\partial x}\left(\frac{\partial\tau_0}{\partial z}\right)^2\frac{\partial\tau_{\theta_2}}{\partial x} \\
&+\left(2\frac{\partial\tau_0}{\partial z}\left((v_-^4 - v_+^4)\left(\frac{\partial\tau_0}{\partial x}\right)^2 + v_+^2\right)\right. \\
&\left.+4(v_-^4 - v_+^4)\left(\frac{\partial\tau_0}{\partial z}\right)^3\right)\frac{\partial\tau_{\theta_2}}{\partial z} \\
&= -\left((v_-^4 - v_+^4)\left(\frac{\partial\tau_0}{\partial z}\right)^2\left(\frac{\partial\tau_\theta}{\partial x}\right)^2\right. \\
&+2(v_-^4 - v_+^4)\frac{\partial\tau_0}{\partial x}\left(\frac{\partial\tau_0}{\partial z}\right)^2\frac{\partial\tau_\theta}{\partial x} \\
&+2(v_-^4 - v_+^4)\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_0}{\partial z}\left(2\frac{\partial\tau_\theta}{\partial x} + \frac{\partial\tau_0}{\partial x}\right)\frac{\partial\tau_\theta}{\partial z} \\
&+\left((v_-^4 - v_+^4)\left(\frac{\partial\tau_0}{\partial x}\right)^2 + v_+^2\right)\left(\frac{\partial\tau_\theta}{\partial z}\right)^2 \\
&+4(v_-^4 - v_+^4)\left(\frac{\partial\tau_0}{\partial z}\right)^3\frac{\partial\tau_\theta}{\partial z} \\
&\left.+6(v_-^4 - v_+^4)\left(\frac{\partial\tau_0}{\partial z}\right)^2\left(\frac{\partial\tau_\theta}{\partial z}\right)^2\right). \quad \text{(A-6)}
\end{aligned}$$

For simplicity, I assume $v_- = 0$, $v_+ = v$, and thus, inserting the trial solution, equation A-3, into equation A-1 yields again a long formula, but by setting $\tan\theta = 0$, I obtain the zeroth-order term given by

$$\left(\frac{\partial\tau_0}{\partial x}\right)^2 + \left(\frac{\partial\tau_0}{\partial z}\right)^2 = \frac{1}{v^2}, \quad \text{(A-7)}$$

which is the conventional eikonal form for waves emanating from a source. By equating the coefficients of the powers of the independent parameter $\tan\theta$, in succession, we end up first with the coefficients of second power in $\tan\theta$, simplified by using equation A-7, and given by

$$v^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_1}{\partial x} + v^2\frac{\partial\tau_0}{\partial z}\frac{\partial\tau_1}{\partial z} = -\frac{1}{2}, \quad \text{(A-8)}$$

which is a first-order linear partial differential equation in τ_θ. The coefficient of $\tan\theta^4$, with some manipulation, has the following form:

$$\begin{aligned}
&v^2\frac{\partial\tau_0}{\partial x}\frac{\partial\tau_2}{\partial x} + v^2\frac{\partial\tau_0}{\partial z}\frac{\partial\tau_2}{\partial z} \\
&= -\frac{1}{2}\left(1 - v^2\left(\frac{\partial\tau_1}{\partial x}\right)^2 - v^2\left(\frac{\partial\tau_1}{\partial z}\right)^2\right), \quad \text{(A-9)}
\end{aligned}$$

which again is a first-order linear partial differential equation in τ_{θ_2} with an obviously more complicated source function given by the right-hand side. Though the equation seems complicated, many of the variables of the source function (right-hand side) can be evaluated during the evaluation of equations A-7 and A-8 in a fashion that will not add much to the cost.

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